$3 + 2 + 1\frac{1}{2} + \frac{1}{2} \rightarrow$ 7p

1 a $L(\dot{\theta}_1, \theta_1, \dot{\theta}_2, \theta_2) = T - V$

$\boxed{3}$ voor kleine uitwijkingen, $\theta_1, \theta_2 \ll 1$.

$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2)$ met x de verticale uitwijking. Voor kleine uitwijkingen geldt:

$$T \simeq \frac{1}{2} m ((l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2) = \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$V = V_{grav} + V_{veer}$

$V_{grav} = mgh_1 + mgh_2$ met h de hoogte gemeten vanaf de evenwichtstand.

$V_{grav} = mg(h_1 + h_2) = mg(l - l\cos\theta_1 + l - l\cos\theta_2) = mgl(2 - \cos\theta_1 - \cos\theta_2)$

$$V_{grav} \simeq mgl(2 - 1 + \tfrac{1}{2}\theta_1^2 - 1 + \tfrac{1}{2}\theta_2^2)$$

$V_{veer} = \frac{1}{2} k (x_1 - x_2)^2$ met $x_1 - x_2$ het verschil tussen de uitwijkingen. Voor kleine uitwijkingen geldt

$$V_{veer} \simeq \frac{1}{2} k (l\theta_1 - l\theta_2)^2 = \frac{1}{2} k l^2 (\theta_1 - \theta_2)^2 = \frac{1}{2} k l^2 (\theta_2 - \theta_1)^2$$

$$L \simeq \frac{1}{2} m l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mgl}{2} (\theta_1^2 + \theta_2^2) - \frac{1}{2} k l^2 (\theta_1 - \theta_2)^2$$

b $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \frac{\partial L}{\partial \theta_1} \Rightarrow ml^2\ddot\theta_1 = -mgl\,\theta_1 - kl^2\theta_1 + kl^2\theta_2$

$$ml^2\ddot\theta_1 + mgl\,\theta_1 + kl^2\theta_1 - kl^2\theta_2 = 0$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = \frac{\partial L}{\partial \theta_2} \Rightarrow ml^2\ddot\theta_2 = -mgl\,\theta_2 - kl^2\theta_2 + kl^2\theta_1$

$\boxed{2}$

$$ml^2\ddot\theta_2 + mgl\,\theta_2 + kl^2\theta_2 - kl^2\theta_1 = 0$$

$$M\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} + C\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}ml^2 & 0\\ 0 & ml^2\end{pmatrix}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} + \begin{pmatrix}mgl+kl^2 & -kl^2\\ -kl^2 & mgl+kl^2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

c) $\theta(t) = \theta_0\cos\omega t = Ae^{i\omega t}$

$\theta_1 = A_1e^{i\omega t}$ $\quad\ddot\theta_1 = -A_1\omega^2e^{i\omega t}$

$\theta_2 = A_2e^{i\omega t}$ $\quad\ddot\theta_2 = -A_2\omega^2e^{i\omega t}$

$$M\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} + C\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}ml^2 & 0\\ 0 & ml^2\end{pmatrix}\begin{pmatrix}-A_1\omega^2e^{i\omega t}\\ -A_2\omega^2e^{i\omega t}\end{pmatrix} + \begin{pmatrix}mgl+kl^2 & -kl^2\\ -kl^2 & mgl+kl^2\end{pmatrix}\begin{pmatrix}A_1e^{i\omega t}\\ A_2e^{i\omega t}\end{pmatrix}$$

$$= \begin{pmatrix}-\omega^2ml^2 & 0\\ 0 & -\omega^2ml^2\end{pmatrix}\begin{pmatrix}A_1e^{i\omega t}\\ A_2e^{i\omega t}\end{pmatrix} + \begin{pmatrix}mgl+kl^2 & -kl^2\\ -kl^2 & mgl+kl^2\end{pmatrix}\begin{pmatrix}A_1e^{i\omega t}\\ A_2e^{i\omega t}\end{pmatrix}$$

$$= \begin{pmatrix}-\omega^2ml + mgl + kl^2 & -kl^2\\ -kl^2 & -\omega^2ml^2 + mgl + kl^2\end{pmatrix}\begin{pmatrix}A_1e^{i\omega t}\\ A_2e^{i\omega t}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\boxed{1\tfrac{1}{2}}/3$

$$\begin{vmatrix}-\omega^2ml + mgl + kl^2 & -kl^2\\ -kl^2 & -\omega^2ml^2 + mgl + kl^2\end{vmatrix} = (-\omega^2ml^2 + mgl + kl^2)^2 - (-kl^2)^2 = 0$$

$$= \omega^4m^2l^4 + m^2g^2l^2 + \cancel{k^2l^4} - 2\omega^2m^2l^3 + 2mgl^3k - 2\omega^2ml^4k - \cancel{k^2l^4} = 0$$

$$= (m^2l^4)\,\omega^4 + (-2m^2l^3 - 2ml^4k)\,\omega^2 + m^2g^2l^2 + 2mgl^3k = 0$$

$$\omega^2 = \frac{2m^2l^3 + 2ml^4k \pm \sqrt{(2m^2l^3 - 2ml^4k)^2 - 4m^2l^4(m^2g^2l^2 + 2mgl^3k)}}{2m^2l^4}$$

$$= \frac{1}{l} + \frac{k}{m} \pm \frac{\sqrt{4m^4l^6 + 4m^2l^8k^2 - 8m^3l^7k - 4m^4l^6g^2 - 8m^3l^7gk}}{2m^2l^4}$$

$$= \frac{1}{l} + \frac{k}{m} \pm \sqrt{\frac{1}{l^2} + \frac{k^2}{m^2} - \frac{2k}{ml} - \frac{g^2}{l^2} - \frac{2gk}{ml}}$$

$$\omega = \pm\sqrt{\frac{1}{l} + \frac{k}{m} + \sqrt{\frac{1}{l^2} + \frac{k^2}{m^2} - \frac{2k}{ml} - \frac{g^2}{l^2} - \frac{2gk}{ml}}}$$

Geen mooie uitkomst, ergens een foutje gemaakt of de benadering $\cos x = 1$ moeten kiezen i.p.v. $\cos x = 1 - \frac{1}{2}x^2$

1c)

$$\begin{vmatrix} -\omega^2 m\ell + k\ell & -k\ell^2 \\ -k\ell^2 & -\omega^2 m\ell^2 + k\ell^2 \end{vmatrix} = \omega^4 m^2 \ell^4 + \cancel{k^2\ell^4} - 2\omega^2 m\ell^2 k - \cancel{k^2\ell^4} = 0$$

$$\omega^2 = \frac{2m\ell^2 k \pm \sqrt{4m^2\ell^4 k^2 \cancel{+ 4m^2\ell^4 \ldots}}}{2m^2\ell^4}$$

$$= \frac{k}{\ell^2} \pm \sqrt{\frac{k^2}{m^2\ell^4}} = \frac{k}{\ell^2} \pm \frac{k}{m\ell^2}$$

$$\omega_1^2 = \frac{k}{\ell^2} + \frac{k}{m\ell^2} = \frac{km + k}{m\ell^2} = \frac{m+1}{\ell^2}\,\frac{k}{m}$$

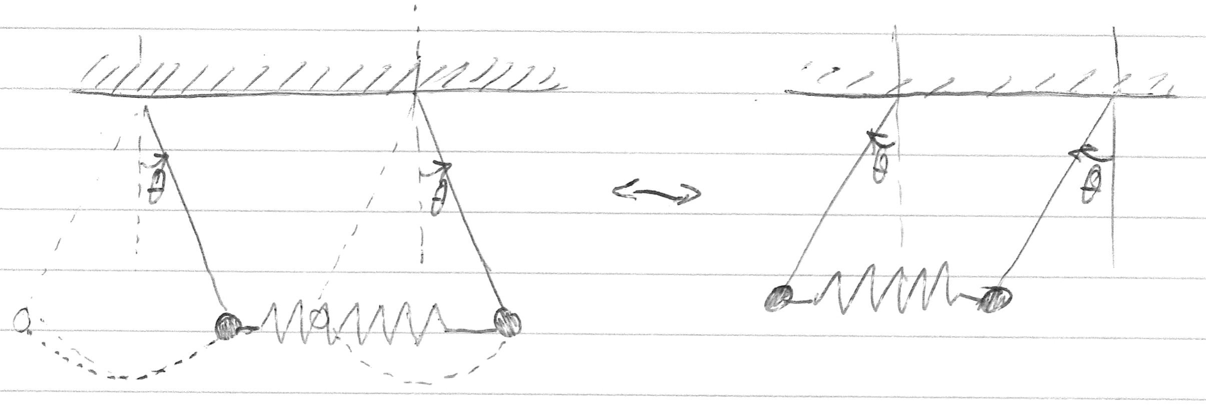
$$\omega_2^2 = \frac{m-1}{\ell^2}\,\frac{k}{m}$$

$$\omega_1 = \pm\sqrt{\frac{m+1}{\ell^2}\,\frac{k}{m}}$$

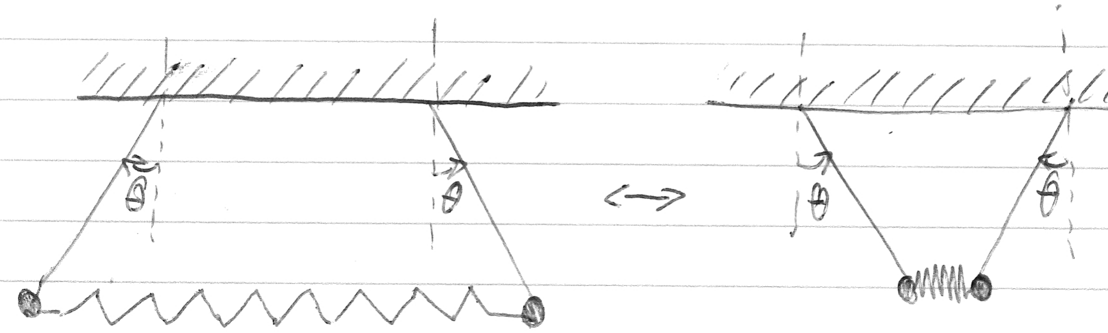
$$\omega_2 = \pm\sqrt{\frac{m-1}{\ell^2}\,\frac{k}{m}}$$

Uitwerking voor $\cos x \approx 1$

1d



$\frac{1}{2}$ | 2



Of een combinatie van de twee hierboven

2a $\hat{x}_i$ $i = 1,2,3$ zijn hoofdassen omdat ze alle drie symmetrie-assen zijn. En symmetrie-assen zijn altijd hoofdassen ✓

$I \equiv I_1 = I_2$ vanwege symmetrie ✓

$I_3 = \frac{1}{2}(\frac{1}{2}m)a^2 + \frac{1}{2}(\frac{1}{2}m)a^2 = \frac{1}{2}ma^2$ 2 schijven met elk een massa van $\frac{1}{2}m$ ✓

$I = \int r^2 \, dm =$ afstand tot 1 schijf is a

Voor iedere schijf afzonderlijk geldt $I = \frac{5}{8}ma^2$ waarom?

Voor 2 schijven wordt dit $I_1 = I_2 = 2 \cdot \frac{5}{8}ma^2 = \frac{5}{4}ma^2$

$1\frac{1}{2}$

b $\underline{\omega} = \omega \hat{\underline{x}}_3$

$\underline{L} = \underline{\underline{I}}\,\underline{\omega}$ $\underline{L} \equiv \int \underline{\tau}\, dt = \int (\underline{r} \times \underline{F})\, dt = \underline{r} \times \int \underline{F}\, dt = \underline{r} \times \underline{P}$
$= \underline{r} \times m\underline{v}$

$r = \begin{pmatrix} 0 \\ -a \\ -a \end{pmatrix}$ $m = m_0$ $v = \begin{pmatrix} 0 \\ 0 \\ v_0 \end{pmatrix}$ $\Rightarrow$ $\underline{L} = \begin{pmatrix} 0 \\ -a \\ -a \end{pmatrix} \times m_0 \begin{pmatrix} 0 \\ 0 \\ v_0 \end{pmatrix}$

$\underline{L} = m_0 \begin{vmatrix} \hat{\underline{x}}_1 & \hat{\underline{x}}_2 & \hat{\underline{x}}_3 \\ 0 & -a & -a \\ 0 & 0 & v_0 \end{vmatrix} = -m_0 a v_0 \hat{\underline{x}}_1 = \begin{pmatrix} -m_0 a v_0 \\ 0 \\ 0 \end{pmatrix}$ ✓

$\underline{\underline{I}} = \begin{pmatrix} \frac{5}{4}ma^2 & 0 & 0 \\ 0 & \frac{5}{4}ma^2 & 0 \\ 0 & 0 & \frac{1}{2}ma^2 \end{pmatrix}$

$\underline{L} = \underline{\underline{I}}\,\underline{\omega} \Rightarrow \begin{pmatrix} -m_0 a v_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{4}ma^2 & 0 & 0 \\ 0 & \frac{5}{4}ma^2 & 0 \\ 0 & 0 & \frac{1}{2}ma^2 \end{pmatrix} \begin{pmatrix} \omega_{x_1} \\ \omega_{x_2} \\ \omega_{x_3} \end{pmatrix}$

$-m_0 a v_0 = \frac{5}{4}ma^2 \omega_x \Rightarrow \omega_{x_1} = \frac{-m_0 a v_0}{\frac{5}{4}ma^2} = -\frac{4m_0 v_0}{5ma}$

$\underline{\omega} = -\frac{4m_0 v_0}{5ma} \hat{\underline{x}}_1$ ✓

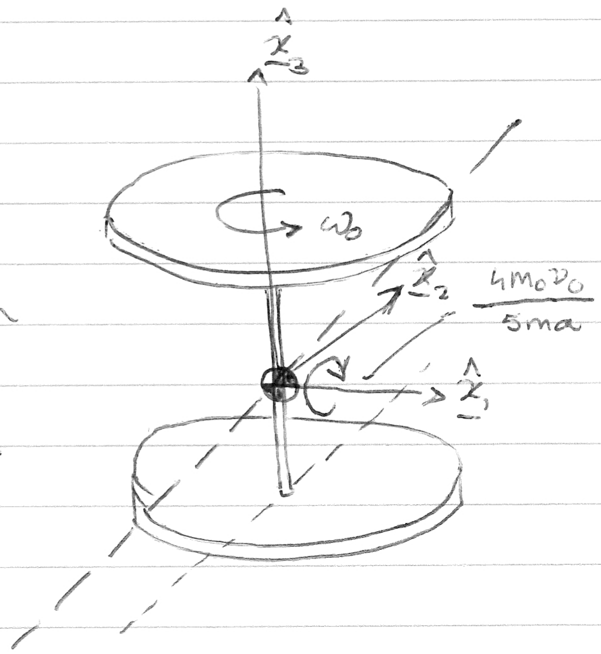
$\underline{\omega} = \begin{pmatrix} -\frac{4m_0 v_0}{5ma} \\ 0 \\ \omega_0 \end{pmatrix}$ is dus de spinvector na de botsing

waar komt deze nu ineens vandaan? 3

c Oriëntatie van de precessie-as

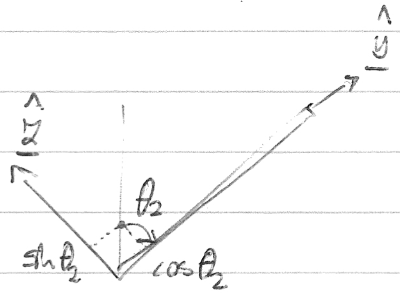
$$\underline{\omega} = \begin{pmatrix} -\frac{4m_0v_0}{5ma} \\ 0 \\ \omega_0 \end{pmatrix}$$

Deze hangt af van m_0 en v_0
De as gaat door het massa-
middelpunt. En de hoek ~~zal~~ van
$\frac{1}{2}$ de as met de verticaal zal
afhangen van de verhouding $\frac{m_0}{m}$,
v_0 en a.



body frame: $\hat{x}_3$
fixed " : $\underline{L}$

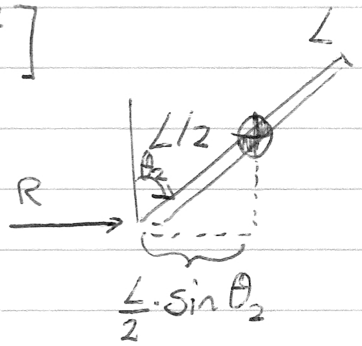
3 a $\omega_x = -\dot{\theta}_2$
$\omega_y = \dot{\theta}_1 \cos\theta_2$
$\omega_z = \dot{\theta}_1 \sin\theta_2$



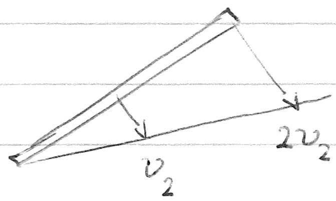
2 $\omega_X = -\dot{\theta}_2 \cos\theta_1$
$\omega_Y = -\dot{\theta}_2 \sin\theta_1$
$\omega_Z = \dot{\theta}_1$

b $E_T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = \frac{1}{2} m ((R + \frac{L}{2}\sin\theta_2)\dot{\theta}_1)^2$
$+ \frac{1}{2} m (\frac{L}{2}\dot{\theta}_2)^2$
$= \frac{1}{2} m \left[(R^2 + \frac{L^2}{4}\sin^2\theta_2 + RL\sin\theta_2)\dot{\theta}_1^2 + \frac{L^2}{4}\dot{\theta}_2^2 \right]$

3

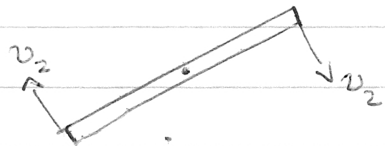


c ~~$E_R = \frac{1}{2} I \omega^2$ met $I = \frac{1}{12} m L^2$~~
~~$E_R = \frac{1}{24} m L^2 \dot{\theta}_2^2 + \dots$~~



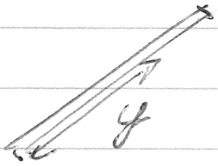
$E_R = \frac{1}{2} I \omega_1^2 + \frac{1}{2} I \omega_2^2$ met $I = \frac{1}{12} m L^2$
$= \frac{1}{2}(\frac{1}{12} m L^2)\, \dot{\theta}_1^2 \sin^2\theta_2 + \frac{1}{2}(\frac{1}{12} m L^2)\, \dot{\theta}_2^2$
$= \frac{m}{2}\left[\frac{1}{12} L^2 \dot{\theta}_1^2 \sin^2\theta_2 + \frac{1}{12} L^2 \dot{\theta}_2^2\right]$

2



$v_2 = \frac{L}{2}\dot{\theta}_2$
$\omega = \frac{v_2}{r} = \dot{\theta}_2$

d $\quad dm = \left(\frac{m}{L}\right) dy$



$$E = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = \frac{1}{2} m \left[(R + y\sin\theta_2)\dot{\theta}_1\right]^2 + \frac{1}{2} m (y\dot{\theta}_2)^2$$

$$= \frac{1}{2} m \left[(R^2 + y^2\sin^2\theta_2 + 2Ry\sin\theta_2)\dot{\theta}_1^2 + y^2\dot{\theta}_2^2\right]$$

$$v = (R + y\sin\theta_2)\dot{\theta}_1 + y\dot{\theta}_2$$

$$E = \int \frac{1}{2}\left[(R^2 + y^2\sin^2\theta_2 + 2Ry\sin\theta_2)\dot{\theta}_1^2 + y^2\dot{\theta}_2^2\right] dm$$

y

$$= \frac{1}{2}\frac{m}{L}\int_0^L \left[R^2 + y^2\sin^2\theta_2 + 2Ry\sin\theta_2)\dot{\theta}_1^2 + y^2\dot{\theta}_2^2 \, dy$$

$$= \frac{m}{2L}\left[R^2\dot{\theta}_1^2 y + \frac{1}{3}y^3\sin^2(\theta_2)\dot{\theta}_1^2 + Ry^2\sin(\theta_2)\dot{\theta}_1^2 + \frac{1}{3}y^3\dot{\theta}_2^2\right]_0^L$$

$$= \frac{m}{2L}\left[R^2\dot{\theta}_1^2 L + \frac{1}{3}L^3\dot{\theta}_1^2\sin^2\theta_2 + RL^2\dot{\theta}_1^2\sin\theta_2 + \frac{1}{3}L^3\dot{\theta}_2^2\right]$$

$$E = \frac{m}{2}\left[R^2\dot{\theta}_1^2 + \frac{1}{3}L^2\dot{\theta}_1^2\sin^2\theta_2 + RL\dot{\theta}_1^2\sin\theta_2 + \frac{1}{3}L^2\dot{\theta}_2^2\right]$$

En $E_T + E_R = \frac{m}{2}\left[R^2\dot{\theta}_1^2 + \frac{1}{4}L^2\dot{\theta}_1^2\sin^2\theta_2 + RL\dot{\theta}_1^2\sin\theta_2 + \frac{1}{4}L^2\dot{\theta}_2^2\right] + \frac{m}{2}\left[\frac{1}{12}L^2\dot{\theta}_2^2 + \frac{1}{12}L^2\dot{\theta}_1^2\sin^2\theta_2\right]$

$$= \frac{m}{2}\left[R^2\dot{\theta}_1^2 + \frac{1}{3}L^2\dot{\theta}_1^2\sin^2\theta_2 + RL\dot{\theta}_1^2\sin\theta_2 + \frac{1}{3}L^2\dot{\theta}_2^2\right]$$

Dit komt overeen met E berekend m.b.v. intergratie